Appendix 1-1

The DALY Formula

The theoretical basis of the DALY construct has been described by Murray (1994, 1996). Murray (1996) presents the DALY formula as:

$DALY_i = YLL_i + YLD_i$ where i is a condition or cause of burden.

The formulas for YLL and YLD presented by Murray (1996) have two additive terms. The first term is with age weighting and the second term is without age weighting. YLL or YLD is a weighted sum of the two terms depending on choice of age weight modulation factor (K). The core of the formula has been considered in each of these terms for the discussion of its derivation. In the following presentation the DALY formula corresponds to core of the YLD formula in Murray's original presentation (Murray 1996). YLL formula can be had simply by setting the disability weight to 1.

Murray presented the DALY formula in 1994 and 1996 slightly differently, as seen below. The formula remains the same, only presentation has changed. One can be obtained by manipulating the other.

$$DALY = -\left[\frac{DCe^{-\beta a}}{(r+\beta)^2} \begin{matrix} e^{-(r+\beta)L}[(r+\beta)(L+a)+1] \\ -[(r+\beta)a+1] \end{matrix} \right] \dots \dots \qquad \dots 1994$$

$$DALY = \frac{DCe^{ra}}{(r+\beta)^2} \left[\begin{matrix} e^{-(r+\beta)(L+a)}[-(r+\beta)(L+a)-1] \\ -e^{-(r+\beta)a}[-(r+\beta)a-1] \end{matrix} \right] \dots \qquad \dots 1996$$

Where;

D = Disability weight[1], C = Age weight constant, β = Age weight parameter,

[1] Demographers use the letter D or d to represent deaths. There is scope for some confusion here, particularly since the YLL is to be calculated for one representative death at first and then multiplied by number of deaths to find total YLL due to premature mortality. In case of premature mortality the disability weight being 1 vanishes from the DALY formula. In that case D can be used to represent deaths without much confusion. In case of YLDs some other letter, say W would have been my preference to represent the disability weight. However I shall continue to use Murray's notation for the sake of consistency with the rest of burden of disease literature.

a = average age at death in case of premature mortality and average age of onset in case of disability, and
L = standard life expectancy at age in case of premature mortality and average duration in case of disability.
r = Discount rate for years to be lived in future.

The DALY formula without the age weighting function i. e. with equal age weights is as follows:

$DALY(r,0) = \frac{D}{r}[1 - e^{-rL}]$ where the notation on the left hand side follows the convention suggested by Murray (1996). All variable are as defined above.

Detailed steps of integration for the DALY formula:

For the sake of easy reference, detailed steps of integration of the age weighting, discounting and duration of illness or years of life lost due to premature mortality are given below to arrive at the above formula.

DALY value of any disability with weight D at any age say a, is given by

$DALY = D \times \text{Age weight} \times \text{Discount factor}$.

The age weight for any age say x is given by the age weight function $Cxe^{-\beta x}$ and the discount factor is given by the function e^{-rx}. To calculate the total DALYs accounted for by a stream of life lost starting from the age at death is simply the integral of the disability weight times the age weight and discount factor. Thus

$$
\begin{aligned}
DALY &= \int_a^{a+L} DCxe^{-\beta x}e^{-r(x-a)}dx \\
&= DC\int_a^{a+L} xe^{-\beta x}e^{-rx+a}dx \\
&= DC\int_a^{a+L} xe^{-\beta x}e^{-rx}e^{ra}dx \\
&= DCe^{ra}\int_a^{a+L} xe^{-\beta x}e^{-rx}dx \\
&= DCe^{ra}\int_a^{a+L} xe^{-\beta x-rx}dx \\
&= DCe^{ra}\int_a^{a+L} xe^{-(r+\beta)x}dx
\end{aligned}
$$

Note that the expression inside the integral is of the form xe^{bx} where $b = -(r+\beta)$. Indefinite integral of this expression is given by $\int xe^{bx}dx = \frac{e^{bx}(bx-1)}{b^2}$ (CRC 1996). Continuing the integration with the above form of anti-derivative for the expression inside the integral:

$$DALY = DCe^{ra}\left[-\frac{x}{(r+\beta)}e^{-(r+\beta)x} - \frac{1}{(r+\beta)^2}e^{-(r+\beta)x} \right]_a^{a+L}$$

$$= DCe^{ra}\left[\begin{array}{c} -\frac{a+L}{(r+\beta)}e^{-(r+\beta)(a+L)} - \frac{1}{(r+\beta)^2}e^{-(r+\beta)(a+L)} \\ +\frac{a}{(r+\beta)}e^{-(r+\beta)a} + \frac{1}{(r+\beta)^2}e^{-(r+\beta)a} \end{array} \right]$$

Multiply, whereever necessary, the numerator and denominator of terms inside the angle bracket with $(r+\beta)$ and factor out $(r+\beta)^2$ from the denominator inside angle brackets to get:

$$DALY = \frac{DCe^{ra}}{(r+\beta)^2}\left[\begin{array}{c} -(a+L)(r+\beta)e^{-(r+\beta)(a+L)} - e^{-(r+\beta)(a+L)} \\ +a(r+\beta)e^{-(r+\beta)a} + e^{-(r+\beta)a} \end{array} \right]$$

$$= \frac{DCe^{ra}}{(r+\beta)^2}\left[\begin{array}{c} e^{-(r+\beta)(L+a)}[-(r+\beta)(L+a)-1] \\ -e^{-(r+\beta)a}[-(r+\beta)a-1] \end{array} \right]$$

This is the 1996 presentation, by Murray, of the DALY formula.
To get the 1994 presentation factor out $e^{-(r+\beta)a}$ from inside the angle brackets.
Note that e^{ra} can be written as $e^{(r+\beta-\beta)a} = e^{(r+\beta)a}e^{-\beta a}$.
So continuing the manipulation of the DALY formula:

$$DALY = \frac{DCe^{ra}e^{-(r+\beta)a}}{(r+\beta)^2}\left[\begin{array}{c} e^{-(r+\beta)L}[-(r+\beta)(L+a)-1] \\ -[-(r+\beta)a-1] \end{array} \right]$$

$$= \frac{DCe^{(r+\beta)a}e^{-\beta a}e^{-(r+\beta)a}}{(r+\beta)^2}$$

$$[e^{-(r+\beta)L}[-(r+\beta)(L+a)-1] + (r+\beta)a + 1]$$

$$= \frac{DCe^{-\beta a}}{(r+\beta)^2}\left[\begin{array}{c} e^{-(r+\beta)L}[-(r+\beta)(L+a)-1] \\ +(r+\beta)a+1 \end{array} \right]$$

$$= -\frac{DCe^{-\beta a}}{(r+\beta)^2}\left[\begin{array}{c} e^{-(r+\beta)L}[1+(r+\beta)(L+a)] \\ -[1+(r+\beta)a] \end{array} \right]$$

This is the 1994 presentation of the same formula.

If there is no age weighting:

$$DALY = \int_a^{a+L} De^{-r(x-a)}dx = De^{ra}\int_a^{a+L} e^{-rx}dx$$
$$= De^{ra}\left[\frac{e^{-rx}}{-r}\Big|_a^{a+L}\right] = -\frac{De^{ra}}{r}[e^{-r(a+L)} - e^{-ra}]$$
$$= -\frac{De^{ra}}{r}[e^{-ra}e^{-rL} - e^{-ra}] = -\frac{D}{r}[e^{-rL} - 1]$$
$$= \frac{D}{r}[1 - e^{-rL}]$$

References:

1. CRC; 1996; Standard mathematical tables and formulae. 30th edition. New York, p391.
2. Murray 1994 and 1996 has been cited in the main text.